On the exact propagator

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Hence (24) predicts that in two dimensions $g = \frac{4}{9}$ while in three dimensions $g = \frac{7}{36}$.

The validity of results (5) and (24) can be tested using numerical data. The numerical data support the relation (5) and support the assumptions made in its derivation. On the other hand, relation (24) is not supported by the numerical data for self-avoiding walks in three dimensions. The numerical work will be published at a later date.

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Domb, C., 1969, *Adv. chem. Phys.*, **15**, 229–59.
Domb, C., Gillis, J., and Wilmers, G., 1965, *Proc. Phys. Soc.*, **85**, 625–45.
Ferer, M., Moore, M. A., and Wortis, M., 1969, *Phys. Rev. Lett.*, **22**, 1382–5.
Fisher, M. E., 1964, *J. math. Phys.*, **5**, 944–62.
—— 1966, *J. chem. Phys.*, **44**, 616–22.
Fisher, M. E., and Burford, R. J., 1967, *Phys. Rev.*, **156**, 583–622.
Hammersley, J. M., 1957, *Proc. Camb. Phil. Soc.*, **53**, 642–5.
—— 1961, *Proc. Camb. Phil. Soc.*, **57**, 516–23.
Martin, J. L., Sykes, M. F., and Hioe, F. T., 1967, *J. chem. Phys.*, **46**, 3478–81.
Ornstein, L. S., and Zernike, F., 1914, *Proc. Acad. Sci. Amsterdam*, **17**, 793.

On the exact propagator

Abstract. We evaluate the propagator for an electron subject to a harmonic force, a constant magnetic field and a time prescribed electric field, exactly. However, our primary concern is to draw attention to important works in the literature that have been overlooked, a result of which has brought some duplication without always corresponding methodological improvement.

We employ functional integration and a result due to Pauli (1952), who evaluated the propagator for a harmonically bound particle under the influence of a time varying force. Pauli's method is based on Van Vleck's work (1928) in connection with the correspondence principle. Extensions of these works appear in a beautiful paper of De Witt (1957) dealing with quantization in curvilinear spaces.

The functional integration methods are mainly Lagrangian based, and the derivation that follows demonstrates the power of Lagrangian quantum mechanics in that an exact propagator can be obtained in cases in which neither energy levels nor eigenfunctions in the configuration representation exist. However, one should not preclude the existence of eigenfunctions, for example, in the momentum representation. An example of this nature is provided in ter Haar (1964) for the propagator of a particle in a constant field of force.

The various particular cases of the present evaluation appeared in a paper by Kennard (1927), who studied their transformation properties. Kennard was the first to obtain the propagator for the free particle, the harmonic oscillator, the particle subject to a constant force, and the charged particle in a constant magnetic field, using a semiclassical approach.

Sondheimer and Wilson (1951) obtained the propagator for an electron in a constant uniform magnetic field in their work on the diamagnetism of free electrons. Their method basically is a direct solution of the appropriate Schrödinger equation.

Apparently the most elegant and far reaching method of derivation is that of functional integration. The above cases and certain of their combinations are found as problems in the book of Feynman and Hibbs (1965). Also the paper of Abé (1954, in japanese) develops a systematic way for the derivation of these propagators. The methods employed in Abé's paper can be found in Papadopoulos (1968) in connection with functional integrals in Brownian motion. Choquard (1955) gives a detailed account of the semiclassical treatment of quantum mechanics using Feynman's representation.

In addition to the literature up to the mid-fifties, re-evaluations of these propagators appear every now and then, with methods which generally involve higher complexity. For example, the propagator for the harmonic oscillator is dealt with in a paper by Davies (1957), using functional integration, but in a rather incomplete manner in the sense that the evaluation of the pre-exponential time dependent factor of the propagator was left out. Lukes and Somaratna (1969) re-evaluate the propagator for a charged particle in a constant electric field, in connection with Stark effect calculations. The method used was a path integral one in phase space. (For certain corrections in this paper the reader is referred to Whitcombe 1971.)

We wish now to employ the Van Vleck–Pauli method for obtaining the propagator of an electron subject to a combination of all the potentials mentioned earlier on.

The Lagrangian for such an electron, when the magnetic field is taken in the z direction, is given by

$$L[\boldsymbol{\xi}] = \frac{m}{2}(\dot{\boldsymbol{\xi}}^2 - \Omega^2\boldsymbol{\xi}^2 + \omega\tilde{\dot{\boldsymbol{\xi}}}_{\perp}\mathbf{J}\boldsymbol{\xi}_{\perp}) + e\boldsymbol{\mathscr{E}}(\tau)\cdot\boldsymbol{\xi} \tag{1}$$

where $\omega = eB/m$ is the cyclotron frequency, Ω the oscillator frequency, and $\boldsymbol{\xi}_{\perp}$ denotes the component of the electron position vector perpendicular to the magnetic field $\boldsymbol{B}$. The term $\mathbf{J}$ is the (2×2) matrix given by

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and its properties and usefulness are given in an earlier work (Papadopoulos and Jones 1971) in connection with the magnetization of conduction electrons in simple metals. There the propagator for an electron in a magnetic field and an impulsive electric field is obtained.

The required propagator is given by the conditional functional integral

$$K(\boldsymbol{x}t|\boldsymbol{x}'0) = \int \exp\left(\frac{\mathrm{i}}{\hbar}\int_0^t L[\boldsymbol{\xi}(\tau)]\,\mathrm{d}\tau\right)\mathscr{D}[\boldsymbol{\xi}] \tag{2}$$

$$\boldsymbol{\xi}(0) = \boldsymbol{x}' \qquad \boldsymbol{\xi}(t) = \boldsymbol{x}$$

where $\mathscr{D}[\boldsymbol{\xi}]$ is the Feynman path differential measure.

Now, since the Lagrangian (1) is quadratic, the functional integral (2) can be evaluated exactly, using the Van Vleck–Pauli result, in terms of the classical action, $S(\boldsymbol{x}t|\boldsymbol{x}'0)$, of an electron starting from $\boldsymbol{x}'$ at time $t = 0$ and reaching $\boldsymbol{x}$ at time t.

We have

$$K(\boldsymbol{x}t|\boldsymbol{x}'0) = \left\{\det\left(\frac{\mathrm{i}}{2\pi\hbar}\frac{\partial^2}{\partial\boldsymbol{x}\partial\boldsymbol{x}'}S(\boldsymbol{x}t|\boldsymbol{x}'0)\right)\right\}^{1/2}\exp\left(\frac{\mathrm{i}}{\hbar}S(\boldsymbol{x}t|\boldsymbol{x}'0)\right). \tag{3}$$

For the details see De Witt (1957).

In our opinion (3) furnishes the simplest and most effective way of obtaining the propagator exactly, when dealing with quadratic Lagrangians.

As is well known the classical action S obeys the Hamilton–Jacobi equation from which it can be obtained, but we prefer to derive it from the equations of motion associated with the Lagrangian (1)

$$\begin{aligned}\ddot{\boldsymbol{\xi}}_\perp+\omega\mathbf{J}\dot{\boldsymbol{\xi}}_\perp+\Omega^2\boldsymbol{\xi}_\perp &= \frac{e}{m}\,{}_\perp\mathscr{E}(\tau)\\ \ddot{\xi}_3+\Omega^2\xi_3 &= \frac{e}{m}\,{}_3\mathscr{E}(\tau)\end{aligned} \tag{4}$$

subject to the end conditions $\boldsymbol{\xi}(0) = \boldsymbol{x}'$, $\boldsymbol{\xi}(t) = \boldsymbol{x}$. The algebra for the solution of the equations of motion (4) can be significantly reduced by employing a method similar to the one used in Papadopoulos (1971) in connection with the magnetization of harmonically bound charges.

We confine ourselves here to writing down the result for the classical action

$$\begin{aligned}S(\boldsymbol{x}t|\boldsymbol{x}'0) &= \int_0^t L[X(\tau)]\,\mathrm{d}\tau\\ &= \frac{m\Omega'}{2\sin\Omega' t}\Bigg[(\boldsymbol{x}_\perp{}^2+\boldsymbol{x}_\perp{}'^2)\cos\Omega' t-2\tilde{\boldsymbol{x}}_\perp{}'\exp\left(\mathbf{J}\frac{\omega}{2}t\right)\boldsymbol{x}_\perp\\ &\quad+\frac{2}{\Omega'}\frac{e}{m}\int_0^t\left\{\tilde{\boldsymbol{x}}_\perp{}'\sin\Omega'(t-\tau)+\tilde{\boldsymbol{x}}_\perp\exp\left(-\mathbf{J}\frac{\omega}{2}t\right)\sin\Omega'\tau\right\}\exp\left(\mathbf{J}\frac{\omega}{2}\tau\right)\\ &\quad\times\mathscr{E}_\perp(\tau)\,\mathrm{d}\tau-\frac{2}{\Omega'^2}\left(\frac{e}{m}\right)^2\int_0^t\int_0^\tau\sin\Omega'(t-\tau)\sin\Omega'\tau'\tilde{\mathscr{E}}_\perp(\tau)\\ &\quad\times\exp\left(-\mathbf{J}\frac{\omega}{2}(\tau-\tau')\right)\mathscr{E}_\perp(\tau')\,\mathrm{d}\tau\,\mathrm{d}\tau'\Bigg]+\frac{m\Omega}{2\sin\Omega t}\Bigg\{(x_3{}^2+x_3{}'^2)\cos\Omega t\\ &\quad-2x_3{}'x_3+\frac{2}{\Omega'}\frac{e}{m}\int_0^t\{x_3{}'\sin\Omega'(t-\tau)+x_3\sin\Omega'\tau\}\mathscr{E}_3(\tau)\,\mathrm{d}\tau\\ &\quad-\frac{2}{\Omega'^2}\left(\frac{e}{m}\right)^2\int_0^t\int_0^\tau\sin\Omega(t-\tau)\sin\Omega\tau'\mathscr{E}_3(\tau)\mathscr{E}_3(\tau')\,\mathrm{d}\tau\,\mathrm{d}\tau'\Bigg\}\end{aligned} \tag{5}$$

where $X(\tau)$ is the classical path between $\boldsymbol{x}'$ and $\boldsymbol{x}$, and $\Omega' = \{\Omega^2+(\omega/2)^2\}^{1/2}$. Using (5) we find the pre-exponential factor in (3) and the required propagator takes the form

$$K(\boldsymbol{x}t|\boldsymbol{x}'0) = \frac{m\Omega'}{2\pi\mathrm{i}\hbar\sin\Omega' t}\left(\frac{m\Omega}{2\pi\mathrm{i}\hbar\sin\Omega t}\right)^{1/2}\exp\left(\frac{\mathrm{i}}{\hbar}S(\boldsymbol{x}t|\boldsymbol{x}'0)\right). \tag{6}$$

It is now a matter of routine exercise to pass to the various limits as Ω, ω, $\mathscr{E}$, or combinations of them and obtain the different cases appearing in the literature.

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Abé, R., 1954, *Busseiron Kenk-yu*, **79**, 101–28.
Choquard, Ph., 1955, *Helv. Phys. Acta*, **28**, 89–157.
Davies, H., 1957, *Proc. Camb. Phil. Soc.*, **53**, 199–205.
Feynman, R. P., and Hibbs, A. R., 1965, *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill), Pp. 63–4.
ter Haar, D., 1964, *Selected Problems in Quantum Mechanics* (London: Infosearch), p. 139.
Kennard, E. H., 1927, *Z. Phys.*, **44**, 326–52.
Lukes, T., and Somaratna, K. T. S., 1969, *J. Phys.* C: *Solid St. Phys.*, **2**, 586–92.
Pauli, W., 1952, *Ausgewälte Kapitel der Feldquantisierung* (*Lecture notes*) (Zürich: ETH), Pp. 139–52.
Papadopoulos, G. J., 1968, *J. Phys.* A: *Gen. Phys.*, **1**, 431–46.
—— 1971, *J. Phys.* A: *Gen. Phys.*, in the press.
Papadopoulos, G. J., and Jones, A. V., 1971, *J. Phys.* F: *Metal Phys.*, **1**, 593–607.
Sondheimer, E. H., and Wilson, A. H., 1951, *Proc. R. Soc.* A, **210**, 173–90.
Van Vleck, J. H., 1928, *Proc. Natn. Acad. Sci.*, **14**, 178–88.
Whitcombe, A. R., 1971, *J. Phys.* C: *Solid St. Phys.*, **4**, 1–2.
de Witt, B. S., 1957, *Rev. mod. Phys.*, **29**, 377–97.

Absolute intensities of cosmic ray muons above 3·48 and 7·12 GeV/c

Abstract. The new Durham spectrograph MARS has been used to determine absolute intensities of cosmic ray muons in the near vertical direction with momenta above 3·48 and 7·12 GeV/c. The intensities are found to be close to those previously reported by Aurela and Wolfendale in 1967, the present intensities being higher by some $(7{\cdot}7 \pm 1{\cdot}3)$% and $(1{\cdot}7 \pm 1{\cdot}4)$%, at the respective momenta. Comparison is also made with the results of other recent measurements.

The momentum spectrum of cosmic ray muons at ground level is a key measurement not only for cosmic ray phenomenology but also because of its relevance to the energetic interactions in the upper levels of the atmosphere, from which the parents of the muons are derived, and to the interpretation of the subsequent behaviour of the muons in their penetration to great depths underground. Absolute intensities rather than spectral shapes are often of importance and where these have not been determined in experiments (usually because of difficulties concerning uncertain edge effects of detectors) it has been customary to normalize the results to a value given by Rossi (1948) at 1 GeV/c. This procedure has been followed by many workers but,